

# Supergroup approach to the Hubbard model

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## Abstract

Based on the revealed hidden supergroup structure, we develop a new approach to the Hubbard model. We reveal a relation of even Hubbard operators to the spinor representation of the generators of the rotation group of four-dimensional spaces. We propose a procedure for constructing a matrix representation of translation generators, yielding a curved space on which dynamic superfields are defined. We construct a new deformed nonlinear superalgebra for the regime of spinless Hubbard model fermions in the case of large on-site repulsion and evaluate the effective functional for spinless fermions.

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# 1 Introduction

The Hubbard model [1] remains the main test ground for investigating the effects of strong correlations between electrons. Traditionally, the effects of a strong Coulomb interaction play an important role in understanding the mechanisms of high-temperature superconductivity, the physics of the Mott–Hubbard-type metal–insulator transition, and the related magnetic states. Experimental evidence of a spin liquid in organic metal (BEDT–TTF)<sub>2</sub>Cu<sub>2</sub>(CN)<sub>3</sub> [2] additionally complicated the situation: it has become necessary to take the spin liquid into account in this model. We believe that this has revitalized the problem of developing new approaches to the Hubbard model based on field theory methods and especially on using the functional integral techniques.

Using the path integral proposed for the Hubbard model in [3], [4], we here continue developing the superfield formulation. We discover a nonlinear hidden supergroup occurring in strongly correlated systems and show how a matrix representation for the coordinate shift generators can be introduced together with the collective dynamic fields. We compute the effective functional based on the new nonlinear superalgebra, also obtained here.

We recall the crucial points of the approach in [3]. As the starting formulation, we take the atomic representation of the Hubbard model. A property of this description is that it allows introducing a local supergroup whose generators are defined by the full set of the on-site Hubbard operators. The supergroup generators act globally and are independent of the lattice coordinates. Next, a supercoherent state is constructed that specifies a superorbit of our supergroup and contains a set of dynamic (i.e., time and coordinate dependent) fermionic and bosonic fields. Passing from the operator to the superfield formulation is achieved using the effective functional of the form

$$\hat{L} = \frac{\langle G | \partial / \partial \tau - H_{\text{Hub}} | G \rangle}{\langle G | G \rangle}, \quad (1)$$

where  $|G\rangle$  is the supercoherent state that specifies the supergroup orbit to be defined below and  $H_{\text{Hub}}$  is an operator expression for the Hubbard model.

We emphasize that one of the fundamental questions of strongly correlated systems is the problem of the existence of a hidden supergroup structure and the procedure for introducing it into the Hubbard model. Identifying this superstructure would allow clarifying the question of new symmetries and supersymmetries and developing a procedure for calculating effective superfield functionals for the strong Coulomb interaction systems described by models that are nonlinear in the dynamic bosonic and fermionic fields.

A central element of the proposed approach is the procedure for identifying even Hubbard operators with local generators of the Lorentz group and other rotation groups of four-dimensional spaces. In strongly correlated systems, we understand the Lorentz group to be a four-dimensional rotation group containing hyperbolic rotations with a “speed of light” constant depending on the characteristic energy, for example, the parameter of the electron band width or the exchange integral. The full set of Hubbard operators is identified with a Lorentz group superextension, which is a subgroup in the superconformal group.

Another important point relates to the introduction of a matrix representation for the group of translations. This subgroup, via a quotient, allows introducing the coordinate space on which the dynamic bosonic and fermionic fields are defined. We introduce the subgroup

of translations using the procedure not of the group affinization, as in the majority of papers (see, e.g., [5]), but of the deformation and contraction of algebras. Our approach is more complicated but seems to be the only one suitable for the Hubbard model. The tricks that we use allow evaluating the effective functional for some deformation of the algebra of spinless fermions, which is nonlinear in the Fermi and Bose generators.

We show how the hidden supergroup structure emerges in the Hubbard model. Any quantum system is characterized by the wave function  $\Psi$ , which is a function of the coordinates  $x$ ,  $y$ , and  $z$  and time  $t$ , i.e.,  $\Psi = \Psi(x, y, z, t)$ . The most widespread groups act on this wave function via operators defined in differential form. For example, one of the widespread continuous groups acting on the wave function is the Poincaré group, which contains the space-time translation operators

$$P_x = -i\frac{\partial}{\partial x}, \quad P_y = -i\frac{\partial}{\partial y}, \quad P_z = -i\frac{\partial}{\partial z}, \quad P_t = i\frac{1}{c}\frac{\partial}{\partial t},$$

the operators of spatial rotations in the  $(x^i, x^j)$  planes

$$L_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right), \quad L_y = i\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right), \quad L_x = i\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right),$$

and the operators of hyperbolic rotations (boosts) in the  $(ct, x^k)$  planes

$$L_1 = i\left(t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\right), \quad L_2 = i\left(t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}\right), \quad L_3 = i\left(t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}\right).$$

The action of a finite group on the wave function is given by the exponential map and has the form

$$e^{E_i P_i + H_j L_j + H_k \Sigma_k} \Psi(x, y, z, t) = e^{H_k \Sigma_k} \Psi(Ax + B), \quad (2)$$

where  $E_i$  and  $H_k$  are small parameters, summation over the indices  $i, j, k = 1, 2, 3$  is understood, the matrix  $A$  defines a vector representation of the rotation group, and the matrix  $B$  represents shifts. We note that the parameters  $H_k$  determine both the matrix  $A$  acting on the coordinates and the elements of the rotation group for the function  $\Psi$  in the spinor basis. The wave function  $\Psi$  is defined in the spinor representation and is acted upon by elements of the group parameterized by the  $H_k$ .

## 2 Hubbard model

The Hubbard model [1], [6] is one of the fundamental models describing systems of strongly interacting electrons in solids. It underlies the description of the band magnetism and superconductivity in strongly correlated metals and the metal-insulator phase transition in solids.

The Hubbard model involves electrons that are nondegenerate with respect to the orbital state and propagate by hops over sites of the crystal lattice with a Coulomb interaction at one site. In the secondary quantization representation, the Hubbard model Hamiltonian has the form

$$\hat{H} = -W \sum_{i,j,\sigma} a_{i\sigma}^+ a_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma}, \quad (3)$$

where  $a_{i\sigma}^+$  and  $a_{i\sigma}$  are Fermi operators of creation and annihilation of an electron with the spin  $\sigma$  at the  $i$ th site,  $n_{i\sigma} = a_{i\sigma}^+ a_{i\sigma}$  is the operator of the number of electrons with a given spin  $\sigma$  at the  $i$ th site, and the spin  $\sigma$  takes two values  $\uparrow (+)$  and  $\downarrow (-)$ .

The Hubbard model involves only three parameters: the matrix element  $W$  of the electron transition to the neighboring site, which is related to the electron band width, the parameter  $U$  of Coulomb repulsion of two electrons at a site  $i$ , and the chemical potential  $\mu$  or the electron concentration  $n$  (the average number of electrons per lattice site).

In what follows, we set  $W = 1$  and consider the half-filled band in the case of strong repulsion. In that regime, the lowest-lying states are described by magnetic fields together with fermionic degrees of freedom. The repulsion parameter practically drops out of the problem.

We are interested in the two-dimensional Hubbard model, and we start investigating it from the atomic limit.

### 3 Hubbard operators

Under the condition of a strong Coulomb repulsion, the Coulomb term in the Hamiltonian is taken as the zeroth approximation. Then the zeroth approximation reduces to a one-site problem and can be solved in the basis of localized atomic functions:

$$|ip\rangle: |i0\rangle, |i+\rangle, |i-\rangle, |i2\rangle, \quad (4)$$

where  $|i0\rangle$  is the state in which the  $i$ th site does not contain an electron,  $|i+\rangle \equiv |i\uparrow\rangle$  is the state with one spin-up electron at the  $i$ th site,  $|i-\rangle \equiv |i\downarrow\rangle$  is the state with one spin-down electron at the  $i$ th site, and  $|i2\rangle = |i\uparrow\downarrow\rangle$  is the state with two electrons at the  $i$ th site, one with spin up and the other with spin down. Any state at the site  $i$  can be represented as a superposition of these localized functions:

$$\psi_i = \alpha_i |i0\rangle + \beta_i |i+\rangle + \gamma_i |i-\rangle + \delta_i |i2\rangle, \quad (5)$$

where  $\alpha_i$ ,  $\beta_i$ ,  $\gamma_i$ , and  $\delta_i$  are coefficients of the atomic basis functions with  $\beta_i$  and  $\gamma_i$  being Grassmann numbers. In what follows, we use matrix elements of the superspinor representation of the local supergroup to represent the coefficients of this decomposition.

Transitions between the atomic basis states are described by  $4 \times 4$  matrices corresponding to the *Hubbard operators*:

$$X_i^{pq} = |ip\rangle \langle iq| = \begin{pmatrix} X_i^{00} & X_i^{0+} & X_i^{0-} & X_i^{02} \\ X_i^{+0} & X_i^{++} & X_i^{+-} & X_i^{+2} \\ X_i^{-0} & X_i^{-+} & X_i^{--} & X_i^{-2} \\ X_i^{20} & X_i^{2+} & X_i^{2-} & X_i^{22} \end{pmatrix}. \quad (6)$$

Each Hubbard operator corresponds to a matrix with the unity at the intersection of the  $p$ th row and the  $q$ th column and with all other elements equal to zero. The Hubbard operators satisfy the relations  $X_i^{pq} X_i^{rs} = \delta_{qr} X_i^{ps}$ , which define their algebra, and the completeness condition  $\sum_p X_i^{pp} = 4$  for the diagonal elements.

The Fermi (f-type) operators are  $X_i^{0+}$ ,  $X_i^{0-}$ ,  $X_i^{+2}$ ,  $X_i^{-2}$ ,  $X_i^{+0}$ ,  $X_i^{-0}$ ,  $X_i^{2+}$ , and  $X_i^{2-}$  because they change the number of electrons at a site by an odd number, 1 or 3. They satisfy the anticommutation relations

$$[X_i^{pq}, X_j^{rs}]_+ = X_i^{pq} X_j^{rs} + X_j^{rs} X_i^{pq} = \delta_{ij}(\delta_{qr} X_i^{ps} + \delta_{qr} X_i^{rq}). \quad (7)$$

The Bose (b-type) operators are  $X_i^{+-}$ ,  $X_i^{-+}$ ,  $X_i^{20}$ , and  $X_i^{02}$ . They change the number of electrons at a site by an even number, 0 or 2. They satisfy the commutation relations

$$[X_i^{pq}, X_j^{rs}]_- = X_i^{pq} X_j^{rs} - X_j^{rs} X_i^{pq} = \delta_{ij}(\delta_{qr} X_i^{ps} - \delta_{sp} X_i^{rq}). \quad (8)$$

This formula also holds if it involves at least one b-type operator or one of the diagonal operators  $X_i^{00}$ ,  $X_i^{++}$ ,  $X_i^{--}$ , or  $X_i^{22}$ .

In terms of the  $X$  operator, the Hubbard model Hamiltonian has the form

$$H = \sum_i \{-\mu X_i^{++} - \mu X_i^{--} + (U - 2\mu) X_i^{22}\} - W \sum_{ij} \{(X_i^{0+} + X_i^{2-})(X_j^{+0} + X_j^{-2}) + (X_i^{-0} + X_i^{2+})(X_j^{0-} + X_j^{+2})\}, \quad (9)$$

where the first sum also includes the chemical potential  $\mu$ . In this representation, the Coulomb term is linear, and the kinetic energy term is given by a bilinear form in the  $X$  operators. In what follows, we use this operator formulation because it takes us precisely to the supergroup construction, which, in our opinion, is the central element of all strongly correlated models.

## 4 Minimal superalgebra in the Hubbard model

We use the construction of a “symmetry tower” introduced for the Hubbard model in [4]. We consider the regime of a half-filled band at strong repulsion such that the energy of the  $|+\rangle$  and  $|-\rangle$  states is lower than the energy of the  $|0\rangle$  and  $|2\rangle$  states by  $U/2$ . Under strong repulsion, it is reasonable to proceed from the so-called spinless superalgebra, which is constructed as follows.

1. We start with the superalgebra of Hubbard operators, denoted by  $S2$ . From the eight Fermi-like Hubbard operators, we form creation and annihilation operators taking two orientations of the spin  $\sigma$  into account:

$$\begin{aligned} a_\uparrow &= X^{0+} + X^{-2}, & a_\uparrow^+ &= X^{+0} + X^{2-}, \\ a_\downarrow &= X^{0-} - X^{+2}, & a_\downarrow^+ &= X^{-0} - X^{2-}. \end{aligned} \quad (10)$$

Their anticommutation relations are

$$\{a_\sigma^+, a_{\sigma'}\}_+ = a_\sigma^+ a_{\sigma'} + a_{\sigma'} a_\sigma^+ = \delta_{\sigma\sigma'}. \quad (11)$$

These operators differ from the Hubbard operators in their (anti)commutation relations and therefore give a new algebra, which together with the density and spin operators describes

excitations in the metallic state. In [4], this algebra was denoted by  $S1$ . It differs from the algebra  $S2$  of Hubbard operators.

2. We introduce spinless operators  $a$  and  $a^+$  as

$$a = \frac{a_{\uparrow} + a_{\downarrow}}{\sqrt{2}}, \quad a^+ = \frac{a_{\uparrow}^+ + a_{\downarrow}^+}{\sqrt{2}}, \quad \rho = \frac{1}{2}a^+a. \quad (12)$$

The operators  $(a^+, a, \rho)$  constitute an algebra, denoted here by  $S0$ , because we have the (anti)commutation relations

$$\{a, a^+\}_+ = \hat{1}, \quad [\rho, a^+] = a^+, \quad [\rho, a] = -a. \quad (13)$$

In a matrix representation of this algebra, we construct a product of the set of dynamic Bose and Fermi fields with the corresponding generators. Just this product enters the nonlinear representation of the supergroup via the exponential map of the form

$$U = \exp \begin{pmatrix} -E_z & \chi & \chi & 0 \\ \chi^* & H_z & H^+ & -\chi \\ \chi^* & H^- & -H_z & \chi \\ 0 & -\chi^* & \chi^* & E_z \end{pmatrix}, \quad (14)$$

where  $E_z$  and  $H^{\pm}$  are field variables and  $\chi$  and  $\chi^*$  are Grassmann variables.

3. We now form a spin vector  $\vec{s} = (s^+, s^-, s_z)$  from the operators introduced in item 1:

$$s^+ = \frac{1}{\sqrt{2}}a_{\uparrow}^+a_{\downarrow}, \quad s^- = \frac{1}{\sqrt{2}}a_{\downarrow}^+a_{\uparrow}, \quad s_z = \frac{1}{2}(a_{\uparrow}^+a_{\uparrow} - a_{\downarrow}^+a_{\downarrow}). \quad (15)$$

The commutation relations for the spin operators are

$$[s^+, s^-]_- = s_z, \quad [s_z, s^+]_- = s^+, \quad [s_z, s^-]_- = -s^-. \quad (16)$$

These operators constitute a spin algebra for the spin 1/2.

The spin operators are given by a  $2 \times 2$  matrix; in the four-dimensional square matrix in (14), the product of the magnetic field vector times the spin vector yields a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_z & H^+ & 0 \\ 0 & H^- & -H_z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. From the operators introduced in item 1, we also form a vector  $\vec{\rho} = (\rho^+, \rho^-, \rho_z)$  of the density operator:

$$\rho^+ = \frac{1}{\sqrt{2}}a_{\uparrow}^+a_{\downarrow}^+, \quad \rho^- = \frac{1}{\sqrt{2}}a_{\downarrow}^+a_{\uparrow}^+, \quad \rho_z = \frac{1}{2}(a_{\uparrow}^+a_{\uparrow}^+ - a_{\downarrow}^+a_{\downarrow}^+). \quad (17)$$

These operators constitute an algebra with the commutation relations

$$[\rho^+, \rho^-]_- = \rho_z, \quad [\rho_z, \rho^+]_- = \rho^+, \quad [\rho_z, \rho^-]_- = -\rho^-. \quad (18)$$

In the matrix representation, this algebra yields the matrix of density fields

$$\begin{pmatrix} E_z & 0 & 0 & E^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E^- & 0 & 0 & -E_z \end{pmatrix}.$$

A second algebra of density operators with the operator components  $\vec{\rho} = (a_{\uparrow}a_{\downarrow}, a_{\downarrow}^{\dagger}a_{\uparrow}^{\dagger}, a_{\downarrow}a_{\downarrow}^{\dagger} - a_{\uparrow}^{\dagger}a_{\uparrow})$  and the same commutation relations is also possible.

We thus formed three algebras from the Fermi-like Hubbard operators: one fermionic spinless and two bosonic algebras. The fermionic algebra describes creation and annihilation of spinless electrons at the sites, the spin algebra is used for taking the interaction of the magnetic moments of electrons into account, and the density bosonic algebra is designed for describing the interaction of density fluctuations, for example, for the analysis of density charge waves.

## 4.1 Deformation and contraction of a Lie algebra

Contraction of a Lie algebra is a limit operation of a Lie algebra deformation. Because these techniques have not been previously used in the Hubbard model, we give the necessary definitions below, referring to [7]–[10].

Let  $G$  be the Lorentz group,  $g$  be the Lie algebra of the Lorentz group, and  $a$  be the subalgebra of the rotation group of the three-dimensional space. Then a contraction of  $g$  gives the Lie algebra of the inhomogeneous rotation group. Indeed, let  $\{c_{ij}^k\}$  be the set of structure constants of the Lie algebra  $g$  in a fixed basis  $e_1, \dots, e_n$ , and let  $A(t)$  be a curve in the group of nondegenerate linear transformations of the  $g$  group space such that  $A(1) = E$ . Let  $e_i(t) = A(t)e_i$  and  $c_{ij}^k(t)$  be structure constants of  $g$  in the basis  $\{e_i(t)\}$ . If the structure constants change as  $t$  varies, then the algebra is said to be deformed. If some of the structure constants vanish as  $t$  tends to zero or infinity, then such a process of changing the algebra is called a contraction [7]–[10].

## 4.2 Local supergroup in the Hubbard model

The Hubbard operators depend on the index parameterizing the coordinate, i.e., they are given by direct products of local copies of a set of generators defined at a selected site. We note that this set of coordinate-independent generators constitutes a global superalgebra (i.e., one defined on the entire space). It can be used to construct the local supergroup in the Hubbard model by an exponential map. The dynamics of the system is then given by the dynamic Fermi and Bose fields in the exponential representation. We consider the spinless case in what follows.

## 5 Even Hubbard operators as generators of the group of four-dimensional rotations

We introduce two triples of operators:  $(X^{00} - X^{22}, X^{02}, X^{20})$  and  $(X^{++} - X^{--}, X^{+-}, X^{-+})$ . The commutators of these operators are expressed only in terms of operators from the same triple. This means that the operators divide into two groups, which commute with each other, i.e., define either a direct product of two  $SU(2)$  groups or the direct product  $SU(2) \otimes SU(1, 1)$  of two subgroups. Naturally, the choice of one of these two types of groups must correspond to a certain choice of the fields.

We note that if the commutation relations in the second subalgebra are chosen such that it becomes the  $su(1, 1)$  algebra, then this leads to a factor  $i$  appearing in front of some of the fields in the functional integral (a Wick rotation), and the group and the space hence become hyperbolic.

If linear combinations (half-sum or half-difference) of the generators of these two groups are taken, then it can be easily verified that their commutation relations coincide with the commutation relations of the four-dimensional Lorentz group [11]. Depending on whether hyperbolic rotations are introduced, this rotation group can be made into the group of rotations of the four-dimensional Euclidean space. We obtain the structure of the commutation relations for the six generators:

$$[L_i, L_j] = i\varepsilon_{ijk}L_k, \quad [L_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = i\varepsilon_{ijk}L_k. \quad (19)$$

Here,  $L_i$ ,  $i = 1, 2, 3$ , specify the spatial rotation generators,  $K_j$ ,  $j = 1, 2, 3$ , are the generators of the time axis rotations, and  $\varepsilon_{ijk}$  is the Levi-Civita tensor. Each of these groups constructed on the given generators is a three-parameter group. Three parameters, multiplication by the rotation generators in the exponential map, are responsible for the rotations, and three are responsible for the hyperbolic rotations (boosts). Hubbard operators are proportional to the following combinations: the spin operators  $s_i = L_i + K_i$ , and the density operators  $\rho_i = L_i - K_i$ ,  $i = 1, 2, 3$ .

We hence conclude that the Bose Hubbard operators can be divided into two subgroups, from which the full set of the Lorentz group generators in a spinor basis can be constructed.

We note that the Lorentz group is a subgroup of the conformal group, which is the largest nonlinear group of the four-dimensional space-time on which the wave function of the system is defined. Thus extending the group of rotations of the four-dimensional space to the conformal and further to the superconformal group, we can construct a local (super)group of the Hubbard model. Because the atomic basis is a central point in practically all of the strongly interacting systems, we have revealed a hidden symmetry in strongly correlated system and indicated a way to introduce local (super)groups for the popular models.

Another important point refers to the introduction of a matrix representation for shift generators. The shift generators are usually given by differential operators (derivatives). In the Hubbard model, just a matrix representation is necessary, different from the one that follows from the affinization procedure [5]. We introduce shift operators in terms of decreasing the dimension of the rotation group and the procedure of deforming the algebra of the Lorentz group generators in the four-dimensional space. This procedure is more complicated than affinization, but we believe it is the only one applicable to the Hubbard model.



We introduce a deformation procedure and then a contraction of the even Hubbard generators. We multiply the generators of hyperbolic rotations by a constant  $R$ ,  $K'_i = RK_i$ ,  $i = 1, 2, 3$ , such that the commutation relations become

$$[K'_i, K'_j] = i \left( \frac{1}{R^2} \right) \varepsilon_{ijk} L_k. \quad (20)$$

In what follows, similarly to [12], we regard  $L_i$  as the generators of rotations in the three-dimensional space and  $K'_i$  as the generators of translations. It follows from (20) that via such a transition from the four-dimensional to the three-dimensional space, we have effected a transition from the group of four-dimensional rotations to the group of inhomogeneous three-dimensional rotations, which contain translations in a curved space. In the limit as  $R \rightarrow \infty$ , we obtain the three-dimensional Poincaré group  $ISO(3)$ , with the algebra of generators

$$[L_i, L_j] = i\varepsilon_{ijk} L_k, \quad [L_i, P_j] = i\varepsilon_{ijk} P_k, \quad [P_i, P_j] = 0, \quad (21)$$

where  $P_i = K'_i$  are generators of translations in flat space. The boost generators thus define translations in a curved space whose radius is defined by the expression  $E^2 = E_z^2 + E^+ E^-$ , and their identification with the generators of translations in the curved space allows introducing the coordinates of the base space by replacing  $(E^+, E^-, E_z)$  with  $(x, y, z)$ . We see in what follows that this expression naturally enters the matrix elements of the supergroup and determines the radius of the space where magnetic dynamic fluctuations occur.

## 6 The group space

The group structure that we have revealed on the even Hubbard generators has a subgroup and a quotient group. Based on this, we can introduce the group of motions specifying the coordinate functions and the functions that are dynamic fields in our problem [3]. We consider the finite continuous group  $G$  depending on the parameters  $E_z, E^+, E^-$  and  $H_z, H^+, H^-$ . In what follows, we use the notation  $E_z, E^+, E^-$  with the understanding that these can be replaced with  $(x, y, z)$  whenever necessary. The parameters  $\{a, b\} = \{E_z, E^+, E^-; H_z, H^+, H^-\}$  can be considered the coordinates of a point in the (3+3)-dimensional group space. Each point of the space is set in correspondence with a transformation from the group  $G$ . The point corresponding to the identity transformation is called the initial point of the space. The initial and an arbitrary point of the space define a vector. Any infinitesimal transformation is expressed in terms of the generators of the quotient group  $X_k$  and the subgroup  $Y_\alpha$ :

$$dG = i(da^k X_k + db^\alpha Y_\alpha), \quad (22)$$

where  $a^k$  and  $b^\alpha$  are parameters of the group,  $k = 1, 2, 3$ , and  $\alpha = 1, 2, 3$ . We again note that the quotient group here specifies the “true” coordinates  $(x, y, z)$ .

The 1-forms are expressed in terms of the group elements as

$$G dG^{-1} = G^{-1} dG = i(\omega^i X_i + \theta^\alpha Y_\alpha), \quad (23)$$

where  $X_i$  is a generator of the quotient group [13].

## 7 Structure equations for the group space

Structure equations for the group space coincide in form with the Maurer–Cartan structure equations for a Riemannian space with zero torsion and a nonzero curvature [14]–[17]:

$$\begin{aligned} d\omega^i &= [\omega^k \omega_k^i] \equiv \omega^k \wedge \omega_k^i, \\ d\omega_i^j &= [\omega_i^k \omega_k^j] + R_{i[kh]}^j [\omega^k \omega^h] \equiv \omega_i^k \wedge \omega_k^j + R_{i[kh]}^j \omega^k \wedge \omega^h. \end{aligned} \quad (24)$$

The forms  $\omega^i$  are components of an infinitesimal shift of the origin of a frame with respect to the frame at a point  $a$ . The forms  $\omega_j^i$  are a change in the components of the frame itself. A transformation from the group  $G$  is a rotation if it belongs to the subgroup  $H$ . The transformations from the subgroup  $H$  leave the origin of the group space fixed and constitute the so-called stationary subgroup of this space. A transformation is a shift if it is generated by an infinitesimal transformation  $\omega^i X_i$ .

A general transformation of the group  $G$  is represented as the product

$$G = K(a)H(b), \quad (25)$$

where  $K(a) = e^{ia^j X_j}$  is a transformation belonging to the left coset class  $G/H$  of the group  $G$  with respect to the subgroup  $H$  and  $H(b) = e^{ib^\alpha Y_\alpha}$  is a transformation belonging to the subgroup  $H$ .

## 8 Cartan forms for the Bose Hubbard operators

We consider vectors  $(x, y, z)$  of the coordinates of the  $SO(3)$  or  $SO(2, 1)$  group space, which in matrix (14) are expressed as components of the electric field  $E_z, E^+, E^-$ , the magnetic field  $H_z(x, y, z), H^+(x, y, z), H^-(x, y, z)$ , and the Grassmann  $\chi(x, y, z)$  and conjugate  $\chi^*(x, y, z)$  fermionic fields as functions of the space–time coordinates  $(x, y, z)$ . We use the local spherical coordinates  $(E, \theta, \varphi)$  and  $(H, \sigma, \vartheta)$  to write the electric and magnetic field vectors:

$$\begin{aligned} E_z &= E \cos \theta, & E^+ &= E \sin \theta \cdot e^{i\varphi}, & E^- &= E \sin \theta \cdot e^{-i\varphi}, \\ H_z &= H \cos \sigma, & H^+ &= H \sin \sigma \cdot e^{i\vartheta}, & H^- &= H \sin \sigma \cdot e^{-i\vartheta}. \end{aligned}$$

Then

$$E^2 = E_z^2 + E^+ E^-, \quad H^2 = H_z^2 + H^+ H^-. \quad (26)$$

We place these fields into a supermatrix and write the exponential map in the form

$$U = \exp \begin{pmatrix} E_z & \chi_1 & \chi_2 & E^+ \\ \chi_1^* & H_z & H^+ & \chi_3 \\ \chi_2^* & H^- & -H_z & \chi_4 \\ E^- & \chi_3^* & \chi_4^* & -E_z \end{pmatrix}. \quad (27)$$

Here and hereafter, in the expression for the exponential map, we do not write the standard minus sign in front of the matrix in the exponent. This sign can easily be compensated at

the final stage by redefining the fields. As a result, we obtain a nonlinear representation for the supergroup that locally performs superrotations of the atomic basis. The problem of calculating the matrix elements of this supergroup arises. We note that this problem is very complicated in the general case. In [18], we could evaluate the chiral representation, when matrix (27) involves only the Grassmann fields marked with an asterisk and bosonic fields. Below, we keep to the following strategy.

We first evaluate the matrix consisting of only the components of the electric field  $E$  and the magnetic field  $H$ , whose *contribution to the effective functional is zero in the fermionic fields*:

$$\begin{aligned} \Omega &= \exp \begin{pmatrix} E_z & 0 & 0 & E^+ \\ 0 & H & H^+ & 0 \\ 0 & H^- & -H_z & 0 \\ E^- & 0 & 0 & -E_z \end{pmatrix} = \\ &= \begin{pmatrix} \cosh E + E_z \frac{\sinh E}{E} & 0 & 0 & E^+ \frac{\sinh E}{E} \\ 0 & \cosh H + H_z \frac{\sinh H}{H} & H^+ \frac{\sinh H}{H} & 0 \\ 0 & H^- \frac{\sinh H}{H} & \cosh H - H_z \frac{\sinh H}{H} & 0 \\ E^- \frac{\sinh E}{E} & 0 & 0 & \cosh E - E_z \frac{\sinh E}{E} \end{pmatrix}. \end{aligned} \quad (28)$$

The Cartan differential form is given by

$$\Omega^{-1} d\Omega = \begin{pmatrix} \omega_{11} & 0 & 0 & \omega_{14} \\ 0 & \omega_{22} & \omega_{23} & 0 \\ 0 & \omega_{32} & \omega_{33} & 0 \\ \omega_{41} & 0 & 0 & \omega_{44} \end{pmatrix} \quad (29)$$

or

$$\begin{aligned} \Omega^{-1} d\Omega &= \omega_{11} X^{00} + \omega_{14} X^{02} + \omega_{41} X^{20} + \omega_{44} X^{22} + \omega_{22} X^{++} + \\ &+ \omega_{23} X^{+-} + \omega_{32} X^{-+} + \omega_{33} X^{--}. \end{aligned} \quad (30)$$

We consider a three-dimensional curved space with the curvature radius defined in terms of the invariant  $E$ . A similar argument can also be presented for the  $H$  vector, which means their independence of the coordinates; we therefore assume that  $E = \text{const}$ ,  $dE = 0$ , and  $H = \text{const}$ ,  $dH = 0$ . This leads to the constraints for the components of the  $E$  and  $H$  vectors

$$dE^2 = 2E_z dE_z + E^+ dE^- + E^- dE^+ = 0, \quad dH^2 = 0, \quad (31)$$

once again indicating the curved nature of both the base space and the space where the dynamic fields exist.

For convenience, we represent the Cartan 1-forms as matrix elements of two  $2 \times 2$  matrices: we have

$$\begin{pmatrix} \omega_{22} & \omega_{23} \\ \omega_{32} & \omega_{33} \end{pmatrix} = \begin{pmatrix} g_1 \frac{\sinh H}{H} dH_z - \frac{\sinh^2 H}{H^2} H^+ dH^- & \frac{\sinh^2 H}{H^2} H^+ dH_z + g_1 \frac{\sinh H}{H} dH^+ \\ -\frac{\sinh^2 H}{H^2} H^- dH_z + g_2 \frac{\sinh H}{H} dH^+ & -g_2 \frac{\sinh H}{H} dH_z - \frac{\sinh^2 H}{H^2} H^- dH^+ \end{pmatrix}, \quad (32)$$

where

$$dH_z = \frac{\partial H_z}{\partial x} dx + \frac{\partial H_z}{\partial y} dy + \frac{\partial H_z}{\partial z} dz$$

and similarly for  $dH^\pm$ ,  $g_2 = \cosh H - H_z \sinh H/H$ , and  $g_1 = \cosh H + H_z \sinh H/H$ . The matrix  $\begin{pmatrix} \omega_{11} & \omega_{14} \\ \omega_{41} & \omega_{44} \end{pmatrix}$  for the electric field components is entirely equivalent to matrix (32) with the replacement  $H_z \rightarrow E_z$ ,  $H^\pm \rightarrow E^\pm$ . In what follows, we use expression (32) to obtain the effective functional for the dynamic magnetic fields.

The Cartan forms for the fermionic Hubbard generators are formed similarly to the bosonic case, with the only difference that odd Grassmann fields and the corresponding differentials are involved and the inverse matrix is calculated differently. In the final analysis, these Cartan 1-forms yield a functional of the fermionic dynamic fields.

## 9 Effective functional

The construction of a “symmetry tower” was proposed in [4]. It is based on the following observation: the variables in strongly correlated systems, being generators of some superalgebras under a variation of the Hubbard repulsion and the chemical potential, constitute a chain of superalgebras ( $S_2, S_1, S_0$ ) that specify excitations of those phases that emerge as the repulsion increases. This chain, called the “symmetry tower,” is characterized by the number of odd generators of the superalgebras and starts with a minimal superalgebra that has two fermionic generators of the algebra of spinless fermions.

In this section, we proceed from this superalgebra and construct its deformation, which is nonlinear in the (super)generators involved. We then evaluate the nonlinear exponential representation of the obtained superalgebra. Next, using formula (1) with Hamiltonian (3), we obtain the effective functional for the fields taking values in the chosen supergroup. We note that the quantum super-Yangian first occurred as an example of a nonlinear superalgebra in the one-dimensional Hubbard model in [19].

We take the expression for the supergroup yielding the supercoherent states  $|G\rangle = U|0\rangle$  for the specified regime observed in the Hubbard model under strong repulsion and half-filling in the form

$$U = \exp \begin{pmatrix} E_z & \chi & \chi & E^+ \\ \chi^* & H_z & H^+ & \chi \\ \chi^* & H^- & -H_z & \chi \\ E^- & \chi^* & \chi^* & -E_z \end{pmatrix}. \quad (33)$$

The generators ( $c^+$ ,  $c$ ,  $c^+c$ , and  $1 - \gamma_5/2$ ) involved here are equal to the corresponding matrices at the dynamic fields in the exponent and form a closed superalgebra of a nonlinear

type. We give matrix expressions for the relevant generators:

$$\begin{aligned}
c^+ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, & c &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
c^+c &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, & \frac{1-\gamma_5}{2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{34}$$

The (anti)commutation relations are

$$\begin{aligned}
\{c^+, c\}_+ &= 2(1 + s^+ + s^-), \\
[c^+, s^+]_- &= c^+ \frac{1-\gamma_5}{2} \frac{1-s_z}{2} - \frac{1-\gamma_5}{2} \frac{1+s_z}{2} c^+, \\
[c^+, s^-]_- &= c^+ \frac{1-\gamma_5}{2} \frac{1+s_z}{2} - \frac{1-\gamma_5}{2} \frac{1-s_z}{2} c^+, \\
[c, s^+]_- &= c \frac{1-\gamma_5}{2} \frac{1-s_z}{2} - \frac{1-\gamma_5}{2} \frac{1+s_z}{2} c, \\
[c, s^-]_- &= c \frac{1-\gamma_5}{2} \frac{1+s_z}{2} - \frac{1-\gamma_5}{2} \frac{1-s_z}{2} c, \\
[c^+c, s^+]_- &= -s_z, \quad [c^+c, s^-]_- = s_z, \quad [c^+c, s_z]_- = s^- - s^+, \\
[c^+, \rho^+] &= c \frac{1-\gamma_5}{2} - \frac{1-\gamma_5}{2} c, \quad [c^+, \rho^-] = 0, \quad [c^+, \rho_z] = c^+, \\
[c, \rho^+] &= 0, \quad [c, \rho^-] = c \frac{1+\gamma_5}{2} - \frac{1+\gamma_5}{2} c, \quad [c, \rho_z] = -c, \\
[c^+c, \rho^+] &= -2\rho^+, \quad [c^+c, \rho^-] = -2\rho^-, \quad [c^+c, \rho_z] = 0.
\end{aligned} \tag{35}$$

We do not take the coefficients of the matrix representation of operators (15) and (17) into account here.

Calculating a nonlinear representation of the above (super)group is cumbersome and quite laborious. The problem of calculating the inverse (super)matrix, which is required for calculating the Cartan (super)differential forms, is especially complicated. In the calculations, we used a computer with 2 Gbytes of RAM and the Mathematica 7.0 symbolic calculation system extended by the superEDC program [20] for working with Grassmann numbers, (super)matrices, and Cartan (super)differential forms. The (super)matrix elements for the matrix  $U$  in (33) and the Cartan 1-forms were evaluated in [21], where all the notation used here was also given. With the stabilized algorithm for working with superfields that take values in (super)matrices, the total time for computing the effective functional given below was about 50 hours.

In calculating functional (1), we used the expansion of the kinetic energy in formula (3) and a representation of the product of operators at the neighboring sites in the form of a series in the lattice constant. In the first order, we obtain

$$\begin{aligned}
a(j)a^+(j+b) &= a(j) \exp\left(b \frac{\partial}{\partial x}\right) a^+(j) = \\
&= a(j) \left[ 1 + (b \cdot \nabla) + \frac{(b \cdot \nabla)^2}{2!} + \dots \right] a^+(j) = \\
&= a(x)a^+(x) + a(x)(b \cdot \nabla)a^+(x) + \dots
\end{aligned} \tag{36}$$

We replace the index  $j$  with the coordinate  $x = bj$  for future use in deriving the representation of the effective functional. We note that the operators  $a$  and  $a^+$  are expressed in terms of the Hubbard generators.

Expanding the exponential in a series using formula (36) and the expression for the Cartan (super)differentials in the first order in the lattice constant, we obtain the effective fermionic functional in the form of the sum

$$\hat{L}_1 = (k_1 + k_2)\chi(r)(b \cdot \nabla)\chi^*(r) + k_3\chi(r)\chi^*(r). \tag{37}$$

This functional is quadratic in the spinless fermionic field and contains the dynamic fields  $H_i$  and a coordinate dependence that gives the metric of the curved space. The coefficients  $k_i$  in (37) are given by

$$\begin{aligned}
k_0 &= EH[f_3 + f_2h_3 + f_2(x+z)], \\
k_1 &= f'_2 + f'_{2h} + 2f'_{1hh}(h_1 + x + y + z) + f'_{hh}[H_z h_2 + z h_2 + (x+y)(H_z + z)], \\
k_2 &= k_0[E^2(f_3 - f_2h_4) + (y+z)(zf_3 - E^2f_2) + (H_z - z)(f_4 - f_3h_4)], \\
k_3 &= k_0\{2(f_4 - h_4f) + (x - y - h_5)(f_3 - h_4f_2) - (y+z)[2f_3 + (x-y)f_2]\}.
\end{aligned} \tag{38}$$

These expression include contributions from the derivatives of the magnetic fields; the exact formula for them will be given elsewhere. The other elements of the condensed notation are presented in the appendix.

Matrix (32) gives differentials of the dynamic fields. We can see from (1) which derivatives of these fields appear. It follows from (1) that the effective functional involves only the expression for  $\omega_{22}$ . With all the foregoing, we obtain the expression for the effective functional of bosonic fields:

$$\hat{L}_0 = -[H^-(b \cdot \nabla H^+) + H^+(b \cdot \nabla H^-) + H_z(b \cdot \nabla H_z)] \frac{\sinh^2 H}{H^2} + H_z(b \cdot \nabla H_z) \frac{\sinh H}{H} \cosh H. \tag{39}$$

If the second-order terms in the lattice constant are kept in the kinetic energy expansion in (36), then an effective functional can be obtained that contains quadratic contributions from derivatives of the magnetic fields. This then gives a nonlinear sigma-model, which is an analogue of the Heisenberg model in the functional integral for the Hubbard model. A functional linear in the derivatives is similar to the functional in the Chern–Simons model, which is natural for our problem.

Finally, the full effective functional for the spinless version of the Hubbard model has the form

$$\hat{L}_{\text{eff}} = \hat{L}_0 + \hat{L}_1. \quad (40)$$

We believe that the regime we have chosen in the Hubbard model is related to the spin-liquid phase revealed in the organic metal (BEDT-TTF)<sub>2</sub>Cu<sub>2</sub>(CN)<sub>3</sub>. This compound requires taking a two-dimensional lattice with a hexagonal symmetry.

## 10 Conclusions

We have offered a further development of the approach formulated in [3], [4]. We showed that the supercoherent state of form (33) defines a nonlinear representation of the minimal (super)extension of the Lorentz group. Continuing this procedure of (super)extension leads to a representation of the superconformal group in the superspinor basis, nonlinear in the fermionic fields. We constructed a new nonlinear matrix representation for the translation generators in the (2+1)-dimensional space-time. It defines a nonlinear superalgebra that enlarges the set of (super)algebras that have been proposed in the “symmetry tower.” We note that after [19], the search for a quantum algebra in the Hubbard model was conducted very vigorously but unfortunately unsuccessfully. We have indicated a way to introduce similar structures in the Hubbard model. Superalgebra (35) contains pairs of odd generators that anticommute with the shift operator (for a spherical space, a shift is equivalent to a rotation), as is appropriate for the standard supersymmetry algebra.

The effective functional of form (40) explicitly contains the coordinates of the  $SO(2,1)$  or  $SO(3)$  group spaces and describes the dynamics of Bose and Fermi fields in a three-dimensional curved space-time. A functional of this type, as far as we know, was first obtained in the Hubbard model.

In relation to this work, we note an entirely new problem for the Hubbard model: calculating the cohomology groups, the second and the third in particular, with the purpose of studying the superalgebras in the “symmetry tower.” This approach opens the possibility of studying the mechanism of the Mott-Hubbard-type metal-insulator transition as a spontaneous violation of the (super)conformal (super)group.

## Appendix

The additional notation in formulas (38) is

$$\begin{aligned}
 f(\alpha) &= \frac{(\sinh(\alpha E))/E - (\sinh(\alpha H))/H}{E^2 - H^2}, & f_{hh} &= \frac{\partial}{\partial(H^2)}[H^2 f(\alpha)]_{\alpha=1}, \\
 f_{1hh} &= \frac{\partial}{\partial\alpha}(f_{hh})_{\alpha=1}, & f_{2h} &= \frac{\partial}{\partial H}(H f_2), \\
 f_2 &= \frac{(\sinh E)/E - (\sinh H)/H}{E^2 - H^2}, & f_3 &= \frac{\cosh E - \cosh H}{E^2 - H^2}, & f_4 &= \frac{E \sinh E - H \sinh H}{E^2 - H^2}, \\
 f'_{hh} &= \frac{\sinh H}{H} f_{hh}, & f'_{1hh} &= \frac{\sinh H}{H} f_{1hh}, & f'_2 &= \frac{\sinh H}{H} f_2, & f'_{2h} &= \frac{\sinh H}{H} f_{2h}, \\
 h_0 &= H^+ - H^-, & h_1 &= H^+ + H^-, & h_2 &= H^- + H_z, \\
 h_3 &= H^- - H_z, & h_4 &= H^+ + H_z, & h_5 &= H^+ - H_z.
 \end{aligned}$$

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